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New Recursive Circular Algorithm for Listing All Permutations

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ABSTRACT

In this paper, a new algorithm for generating permutation based on the starter sets under circular operations is presented. Once the starter sets are obtained, the circular and reverse of circular operation are employed to produce distinct permutation from each starter sets.

Keywords: Algorithm, circular, permutation, starter sets

INTRODUCTION

The generation of all $n!$ permutations of n elements is a fundamental problem in combinatorics and important in computing. Various methods on listing all permutations have been published and can be classified into two categories: (i) exchange based techniques; (ii) non-exchange based techniques (Sedgeweck, 1977). The exchange based techniques generate the new permutation by making possible changes among two consecutive elements such as transposition of non adjacent elements (Well, 1961; Heap, 1963), and transposition with adjacent element (Trotter,1962; Johnson, 1963; Ives, 1976). Whilst non-exchange based techniques generate the new permutation with certain restriction such as lexicographic order (Ord-Smith, 1970), and nested cyclic (Langdon, 1976). All these techniques were nicely summarized and surveyed by Sedgeweck (1977).

According to Sedgeweck (1977), generating permutation under cycling restriction is simpler and more powerful compared to others restriction techniques. Langdon (1967) and Iyer (1995) proposed a cycling technique for permutation generation based on 'pivot' permutation. Meanwhile Ibrahim *et al.* (2010) introduced a new permutation technique based on distinct starter sets by employing circular and reversing operations. Their crucial task is to generate the distinct starter sets by eliminating the equivalence starter sets. Although this technique is simple and easy to use, unfortunately eliminating the equivalence starters is quite tedious when the number of elements increases. This paper attempts to overcome this drawback by introducing a new strategy for generating distinct starter sets without eliminating the equivalence starter sets.

MATERIALS AND METHODS

Preliminary Definition

The following definitions will be used throughout this paper.

Definition 1. A *starter set* is a set that is used as a basis to enumerate other permutations.

Definition 2. The *reverse set* is a set that is produced by reversing the order of permutation set.

Definition 3. A l*atin square* of order n is an n× n array in which n distinct symbols are arranged where each symbol occurs once in each row and column.

Definition 4. The *circular permutation* (CP) of order n is a latin square of order n.

Definition 5. The *reverse of circular permutation* (RoCP) is also a latin square of order n which is obtained by reversing arrangement element in each row of circular permutation.

The Development of the Algorithm

The general algorithm for permutation generation follows:
Let S be the set of n elements i.e $(1, 2, 3, \ldots, n\text{-}3, n\text{-}2, n\text{-}1, n)$

Step 1: Set $(1, 2, 3, 4, \ldots, n\text{-}3, n\text{-}2, n\text{-}1, n)$ as initial permutation and without loss of generality, the first elements is fixed.

Step 2: Identify the last three elements of initial permutation from step 1. By employing CP to the last three elements on initial permutation from step 1 will produce other three distinct starter sets:

$1, 2, \ldots, n\text{-}3, n\text{-}2, n\text{-}1, n$
$1, 2, \ldots, n\text{-}3, n\text{-}1, n, n\text{-}2$
$1, 2, \ldots, n\text{-}3, n, n\text{-}2, n\text{-}1$

Step 3: Identify the last four elements of each starter sets in step 2. By employing CP to last four elements on each starter sets in step 2, the 12 distinct starter sets are obtained.

$1, 2, \ldots, n\text{-}3, n\text{-}2, n\text{-}1, n$	$1, 2, \ldots, n\text{-}3, n\text{-}1, n, n\text{-}2$	$1, 2, \ldots, n\text{-}3, n, n\text{-}2, n\text{-}1$
$1, 2,\ldots, n\text{-}2, n\text{-}1, n, n\text{-}3$	$1, 2, \ldots, n\text{-}1, n, n\text{-}2, n\text{-}3$	$1, 2, \ldots, n, n\text{-}2, n\text{-}1, n\text{-}3$
$1, 2, \ldots, n\text{-}1, n, n\text{-}3, n\text{-}2$	$1, 2, \ldots, n, n\text{-}2, n\text{-}3, n\text{-}1$	$1, 2, \ldots, n\text{-}2, n\text{-}1, n\text{-}3, n$
$1, 2, \ldots, n, n\text{-}3, n\text{-}2, n\text{-}1$	$1, 2, \ldots, n\text{-}2, n\text{-}3, n\text{-}1, n$	$1, 2, \ldots, n\text{-}1, n\text{-}3, n, n\text{-}2$

Step n-2: Identify the last $(n - 1)$ elements of each starter sets in step $(n - 3)$. By employing to the last $(n - 1)$ elements on each starter set in step $(n - 3)$, the $\frac{(n-1)!}{2}$ distinct starter sets are obtained.

Step n-1: Perform CP and RoCP simultaneously to all n elements of $\frac{(n-1)!}{2}$ distinct starter sets and $n!$ distinct permutations are obtained.

Step n: Display all $n!$ permutation.
There are $(n\text{-}2)$ steps needed to generate starter set. Then CP and RoCP are employed on these starter sets to list down all $n!$ distinct permutation.

To illustrate this algorithm, let's consider the set of five elements, i.e $S = (1, 2, 3, 4, 5)$

Step 1: Set (1, 2, 3, 4, 5) as initial permutation and without loss of generality, the first element is fixed.

Step 2: Identify the last three elements of initial permutation from step 1. By employing CP to the last three elements on initial permutation from step 1 will produce other three distinct starter sets as shown below:

1, 2, 3, 4, 5
1, 2, 4, 5, 3
1, 2, 5, 3, 4

Step 3: Identify the last four elements of each starter sets in step 2. By employing CP to last four elements on each starter sets in step 2, the 12 distinct starter sets are obtained as shown below:

1, 2, 4, 5, 3	**1, 2, 5, 3, 4**	**1, 2, 3, 4, 5**
1, 4, 5, 3, 2	1, 5, 3, 4, 2	1, 3, 4, 5, 2
1, 5, 3, 2, 4	1, 3, 4, 2, 5	1, 4, 5, 2, 3
1, 3, 2, 4, 5	1, 4, 2, 5, 3	1, 5, 2, 3, 4

Step 4: Perform CP and RoCP simultaneously to all n elements of 12 distinct starter sets and 5! distinct permutations are obtained (see Table I) .

Step 5: Display 5! distinct permutations (see Table I)

Table 1: The 5! distinct permutations

CP	RoCP	CP	RoCP
14532	23541	**14253**	35241
45321	12354	42531	13524
53214	41235	25314	41352
32145	54123	53142	24135
21453	35412	31425	52413
15324	42351	**12534**	43521
53241	14235	25341	14352
32415	51423	53412	21435
24153	35142	34125	52143
41532	23514	41253	35214
13245	54231	**13452**	25431
32451	15423	34521	12543
24513	31542	45213	31254
45132	23154	52134	43125
51324	42315	21345	54312
12453	35421	**14523**	32541
24531	13542	45231	13254
45312	21354	52314	41325
53124	42135	23145	54132
31245	54213	31452	25413
15342	24351	**15234**	43251
53421	12435	52341	14325
34215	51243	23415	51432
42153	35124	34152	25143
21534	43512	41523	32514
13425	52431	**12345**	54321
34251	51243	23451	15432
42513	31524	34512	21543
25134	43152	45123	32154
51342	24315	51234	43215

Remark 2: The bold mark of the permutation represent 12 starter sets for case n = 5.

RESULTS AND DISCUSSION

Some Theoretical Results

The following lemmas and theorem are produced from the recursive circular permutation generation method.

Lemma 1. $2n$ distinct permutations are produced by each distinct starters set.

Proof: Suppose we have a starter set of $A = (1, 2, 3, ... n - 1, n)$ with n distinct elements. By using definition 4 where the all element is cycle to the left, n distinct permutation are obtained as follows:

$$\begin{array}{cccccc}
1 & 2 & \dots & n\text{-}2 & n\text{-}1 & n \\
2 & \dots & n\text{-}2 & n\text{-}1 & n & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n\text{-}2 & n\text{-}1 & n & 1 & 2 & \dots \\
n\text{-}1 & n & 1 & 2 & \dots & \text{n-2} \\
n & 1 & 2 & \dots & n\text{-}2 & n\text{-}1
\end{array}$$

Then from definition 5, reversing each row of CP will also produced other n distinct permutations as follows:

$$\begin{array}{cccccc}
n & n\text{-}1 & n\text{-}2 & \dots & 2 & 1 \\
1 & n & n\text{-}1 & n\text{-}2 & \dots & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\dots & 2 & 1 & n & n\text{-}1 & n\text{-}2 \\
n\text{-}2 & \dots & 2 & 1 & n & n\text{-}1 \\
n\text{-}1 & n\text{-}2 & \dots & 2 & 1 & n
\end{array}$$

Thus $2n$ distinct permutations are produced.

Lemma 2. There are $\frac{(n-1)!}{2}$ distinct starter sets which are generated recursively for $n \geq 3$ under circular operation.

Proof: Let (1, 2, 3, ...,) as initial starter for any $n \geq 3$. By employing CP to the last three elements, three distinct starters are produced, as shown below:

$$\begin{array}{cccccccl}
1 & 2 & 3 & \dots & n\text{-}3 & n\text{-}2 & n\text{-}1 & n \text{ (starter 1)} \\
1 & 2 & 3 & \dots & n\text{-}3 & n\text{-}1 & n & n\text{-}2 \text{ (starter 2)} \\
1 & 2 & 3 & \dots & n\text{-}3 & n & n\text{ -}2 & n\text{-}1 \text{ (starter 3)}
\end{array}$$

Then for each previous starter sets, the last four elements will be selected and by employing CP on these elements of previous starter sets, four distinct starters are produced, as shown below:

From starter 1,

$$\begin{array}{cccccccc}
1 & 2 & 3 & \dots & n\text{-}2 & n\text{-}1 & n & n\text{-}3 \\
1 & 2 & 3 & \dots & n\text{-}1 & n & n\text{-}3 & n\text{-}2 \\
1 & 2 & 3 & \dots & n & n\text{-}3 & n\text{-}2 & n\text{-}1 \\
1 & 2 & 3 & \dots & n\text{-}3 & n\text{-}2 & n\text{-}1 & n
\end{array}$$

From starter 2,

$$\begin{matrix} 1 & 2 & 3 & \dots & n\text{-}1 & n & n\text{-}2 & n\text{-}3 \\ 1 & 2 & 3 & \dots & n & n\text{-}2 & n\text{-}3 & n\text{-}1 \\ 1 & 2 & 3 & \dots & n\text{-}2 & n\text{-}3 & n\text{-}1 & n \\ 1 & 2 & 3 & \dots & n\text{-}3 & n\text{-}1 & n & n\text{-}2 \end{matrix}$$

From starter 3,

$$\begin{matrix} 1 & 2 & 3 & \dots & n & n\text{-}2 & n\text{-}1 & n\text{-}3 \\ 1 & 2 & 3 & \dots & n\text{-}2 & n\text{-}1 & n\text{-}3 & n \\ 1 & 2 & 3 & \dots & n\text{-}1 & n\text{-}3 & n & n\text{-}2 \\ 1 & 2 & 3 & \dots & n\text{-}3 & n & n\text{-}2 & n\text{-}1 \end{matrix}$$

Thus at this stage, the total starter sets are 3 x 4 = 12. The process will be continued until to the last (n - 1) elements is selected.

$$\begin{matrix} 3 & \text{last elements} & \rightarrow & 3 \text{ starter sets} \\ 4 & \text{last elements} & \rightarrow & 4 \text{ starter sets} \\ 5 & \text{last elements} & \rightarrow & 5 \text{ starter sets} \\ & \vdots & \rightarrow & \vdots \\ (n\text{-}2) & \text{last elements} & \rightarrow & (n\text{-}2) \text{ starter sets} \\ (n\text{-}1) & \text{last elements} & \rightarrow & (n\text{-}1) \text{ starter sets} \end{matrix}$$

By product rule, the numbers of the starter sets are

$$3 \text{ x } 4 \text{ x } 5 \text{ x } \dots \text{ x } n \text{ -2 x } n\text{-1} \tag{1}$$

$$= \frac{1}{2} \text{ x } 2 \text{ x } 3 \text{ x } 4 \text{ x } 5 \text{ x } \dots \text{ x } n \text{ -2 x } n\text{-1} \tag{2}$$

$$= \frac{1}{2}\ (n - 1) \tag{3}$$

Remark 3: For case is impossible since it has only one distinct starter set

while $\quad \frac{(2-1)!}{2} = \frac{1}{2}.$

Theorem 1. The generation of all distinct permutation can be obtained by $\frac{(n-1)!}{2}$ distinct starter sets.

Proof: From lemma 2 , there are $\frac{(n-1)!}{2}$ distinct starter for $n \geq 3$ Then from lemma 1, $2n$ distinct permutation are obtained by employing circular and reversing process on the starter sets.

Thus $\frac{(n-1)!}{2}$ X $2n = n!$ permutations are generated.

CONCLUSIONS

Novel approach to the listing n! permutations based on recursive circular generated starter sets is proposed. Furthermore this recursive circular algorithm is efficient since the starter sets can be generated without eliminating the equivalence starter sets, and thus reduces the computation time.

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